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RINGS & FIELDS

Ring:

Let R be a non-empty set and $+$, $\bullet$ be two binary operations in R . $(R, +, \bullet)$ is said to be a ring, if (i) $(R, +)$ is a commutative group, (ii) $(R, \bullet)$ is a semigroup and (iii) Distributive laws hold.

Ring with unity:

In a ring $(R, +, \bullet)$ if there exists $1 \in R$ such that $a \bullet 1 = 1 \bullet a = a$ for every $a \in R$ then we say that R is a ring with unit element or unity element.

Commutative Ring:

In a ring $(R, +, \bullet)$ if $a \bullet b = b \bullet a$ for $a, b \in R$ then we say that R is a commutative ring.

Boolean Ring:

In a ring R if $a^2 = a \forall a \in R$ then R is called a Boolean ring.

Zero divisors of a Ring:

In a ring R , $a \neq 0 \in R$ is said to be a zero divisor if there exists a ' b ' $\neq 0 \in R$ so that $ab = 0$.

A ring R is said to have zero divisors if there exist $a, b \in R$ so that $a \neq 0, b \neq 0$ and $ab = 0$.

A ring R is said to have no zero divisors if there exist $a, b \in R$ and $ab = 0 \Rightarrow a = 0$ or $b = 0$.

Cancellation laws in a Ring:

In a ring R , for $a, b, c \in R$ if $a \neq 0, ab = ac \Rightarrow b = c$ and $a \neq 0, ba = ca \Rightarrow b = c$ then we say that cancellation laws hold in R .

Integral Domain:

A ring is called an integral domain if (i) it is commutative and (ii) it is without zero divisors.

Division Ring or Skew Field:

A ring with at least two elements is called a Division Ring if (i) it has unit element and (ii) all non-zero elements of it are invertible under multiplication.

Field:

A ring with at least two elements is called a Field, if (i) it is commutative (ii) it has unit element and (iii) all non-zero element of it are invertible under multiplication.

A commutative division ring is a field.

Idempotent Element of a Ring:

In a ring R , if $a^2 = a$ for $a \in R$ then 'a' is called idempotent element of R with respect to multiplication.

Nilpotent Element:

Let R be a ring and $a \neq 0 \in R$. If there exists $n \in \mathbb{N}$ such that $a^n = 0$ then 'a' is called nilpotent element of R .

Characteristic of a Ring:

The characteristic of a ring R is defined as the least positive integer p such that $pa = 0$ for all $a \in R$. In case such a positive integer p does not exist then we say that the characteristic of R is zero or infinite.

Note:

- A commutative ring R is an integral domain if and only if the cancellation laws hold in R .
- A division ring has no zero divisors.
- A field has no zero divisors.
- Every field is an integral domain.
- A finite integral domain is a field. If p is a prime then $\mathbb{Z}_p$, the ring of integers modulo p , is a field.
- An integral domain has no nilpotent element other than zero.
- The characteristic of an integral domain is either a prime or zero.
- The characteristic of a field is either a prime or zero.
- The characteristic of a Boolean ring is 2.

Subrings, Ideals and Quotient Ring

Subring:

If $(R, +, \bullet)$ be a ring and S be a non-empty subset of R . If $(S, +, \bullet)$ is also a ring with respect to the two operations in R then we say that S is a subring of R .

Subfield:

Let $(F, +, \bullet)$ be a field and $(S, +, \bullet)$ be a subring of F . If $(S, +, \bullet)$ is a field then we say that S is a Subfield of F .

Note:

- Let $(R, +, \bullet)$ be a ring and $0 \leq r$ be the zero element of R . Then $S = \{0\}$ is a non-empty subset of R so that $(S, +, \bullet)$ is itself a ring. Therefore $(S, +, \bullet)$ is a subring of R .
($\{0\}, +, \bullet$) and $(R, +, \bullet)$ are called trivial or improper subrings of R .
- Let S be a non-empty subset of a ring R . Then S is a subring. Then S is a subring of R if and only if $a - b \in S$ and $ab \in S$ for all $a, b \in S$.
- Let K be a non-empty subset of a field F . Then K is a subfield of F if and only if $a, b \in K \Rightarrow a - b \in K$ and $a \in K, b \neq 0 \Rightarrow ab^{-1} \in K$.
- The intersection of two subrings of a ring r is a subring of R .
- If S_1 and S_2 are two subrings of a ring R then $S_1 \cup S_2$ is a subring of r if and only if $S_1 \subset S_2$ or $S_2 \subset S_1$.
- If R is a ring and $C(R) = \{ x \in R / xa = ax \forall a \in R \}$. Then $C(R)$ is a subring of R . $C(R)$ is called the centre of the ring R .

Ideals:

Let $(R, +, \bullet)$ be a ring. A non-empty subset U of R is called a right ideal if (i) $a, b \in U \Rightarrow a - b \in U$ and (ii) $a \in U$ and $r \in R \Rightarrow ar \in U$.

Let $(R, +, \bullet)$ be a ring. A non-empty subset U of R is called a left ideal if (i) $a, b \in U \Rightarrow a - b \in U$ and (ii) $a \in U$ and $r \in R \Rightarrow ra \in U$.

Let $(R, +, \bullet)$ be a ring. A non-empty subset U of R is called a right ideal if (i) $a, b \in U \Rightarrow a - b \in U$ and (ii) $a \in U$ and $r \in R \Rightarrow ra, ar \in U$.

Ideal Generated by a subset of Ring:

Let S be a subset of a ring R . An ideal U of the ring is said to be generated by S if (i) $S \subset U$ and (ii) for any ideal V of R , $S \subset V \Rightarrow U \subset V$.

The ideal u generated by S is denoted by (S) or $\langle S \rangle$.

Principal Ideal:

An ideal U of a ring R is said to be a principal ideal of R if there exists $a \in R$ such that for any ideal V of R , $a \in V \Rightarrow U \subset V$.

The principal ideal U of the ring R is the ideal of R generated by the element $a \in R$ and is denoted by (a) or $\langle a \rangle$.

The null ideal of a ring is the principal ideal generated by the zero element of the ring. The unit ideal of a ring is the principal ideal generated by the unit element of the ring.

A field has no proper ideals, every ideal of a field is a principal ideal.

Principal ideal Ring:

A ring R is called a principal ideal ring if every ideal in R is a principal ideal.

Every field is a principal ideal ring.

Note:

- If R is a ring then the null ideal $U = \{0\}$ and the unit ideal R are called improper ideals of R .
- Any other ideal of R is called a proper ideal of R .
- If U is an ideal of a ring R with unit element and $1 \in U$ then $U = R$.
- A field has no proper ideal.
- If R is a commutative ring and $a \in R$ then $Ra = \{ ra / r \in R \}$ is an ideal of R .
- The intersection of two ideals of a ring R is an ideal of R .
- If U_1 and U_2 are two ideals of a ring R then $U_1 \cup U_2$ is an ideal of R if and only if $U_1 \subset U_2$ or $U_2 \subset U_1$.
- If U_1 and U_2 are two ideals of a ring R then $U_1 + U_2 = \{ x + y / x \in U_1, y \in U_2 \}$ is also an ideal of R .
- If U_1 and U_2 are two ideals of a ring R then $U_1 + U_2 = (U_1 + U_2)$, the ideal generated by $U_1 \cup U_2$.
- If R is a commutative ring with unit element and $a \in R$, then the $U = \{ ra / r \in R \}$ is a principal ideal of R generated by the element 'a'.
- The ring of integers is a principal ideal ring.

Quotient Ring:

Let R be a ring and U be an ideal of R . Then the set $R/U = \{x + U / x \in R\}$ with respect to addition and multiplication of cosets defined by $(a+U) + (b+U) = (a+b) + U$ and $(a+U) \bullet (b+U) =$

$a+U$ for $a+U, b+U \in R/U$ is a ring. This ring $(R/U, +, \bullet)$ is called the quotient ring or factor ring or residue class ring.

Prime Ideal:

An ideal U of a commutative ring R is said to be prime ideal if for all $a, b \in R$ and $ab \in U \Rightarrow a \in U$ or $b \in U$.

Maximal Ideal:

Let R be a ring and M be an ideal of R , so that $M \neq R$. M is said to be a maximal ideal of R if whenever U ideal of R such that $M \subset U \subset R$ then either $R = U$ or $U = M$.

Note:

- If R/U is the quotient ring then (i) R/U is commutative if R is commutative and (ii) R/U has unity element if R has unity element.
- An ideal U of a commutative ring R , is a prime ideal if and only if R/U is an integral domain.
- In the ring Z of integers, the ideal generated by prime integer is a maximal ideal.
- If M is a maximal ideal of the ring of integers Z then M is generated by prime integer.
- An ideal U of a commutative ring R with unity is maximal if and only if the quotient ring R/U is a field.
- For a commutative ring R with unity, an ideal U is maximal if and only if U is prime ideal.

Homomorphism

Def: Let R, R' be two rings. A mapping $f: R \rightarrow R'$ is said to be a homomorphism if (i) $f(a+b) = f(a) + f(b)$ and (ii) $f(ab) = f(a)f(b)$ for all $a, b \in R$.

Def: If $f: R \rightarrow R'$ is a homomorphism of a ring R into R' then the image set $f(R) = \{ f(x) / x \in R \}$ is called the f homomorphic image of R .

Def: Let R, R' be two rings. A mapping $f: R \rightarrow R'$ is called an epimorphism or onto homomorphism if f is onto mapping.

Def: A homomorphism $f: R \rightarrow R'$ is called a monomorphism if f is one-one mapping.

Def: A homomorphism $f: R \rightarrow R'$ is called an isomorphism if f is both one-one and onto mapping.

Def: A homomorphism $f: R \rightarrow R$ of a ring R into itself is called an endomorphism.

Def: A homomorphism $f: R \rightarrow R$ which is both one-one and onto is called an automorphism.

Notation: If $f: R \rightarrow R'$ is an onto homomorphism or epimorphism then R' is the homomorphic image of R and we write $R \cong R'$.

If $f: R \rightarrow R'$ is an isomorphism then we say that R is isomorphic to R' or R, R' are isomorphic and we write $R \cong R'$.

Natural Homomorphism: If U is an ideal of the ring R , then R/U is also a ring w.r.t. addition and multiplication of cosets. Then the mapping $f: R \rightarrow R/U$ defined by $f(x) = x + U$ for all $x \in R$ called the natural homomorphism from r onto R/U .

Kernel of Homomorphism:

Let R, R' be two rings and $f: R \rightarrow R'$ be a homomorphism. The set $\{x \in R / f(x) = 0'\}$ where $0' \in R'$ is the zero element is defined as the kernel of the homomorphism f and is denoted by $\text{Ker } f$ or $I(f)$.

Note:

- Let $f: R \rightarrow R'$ be a homomorphism of a ring R into the ring R' and $0 \in R, 0' \in R'$ be the zero elements. Then (i) $f(0) = 0'$ (ii) $f(-a) = -f(a) \forall a \in R$ (iii) $f(a - b) = f(a) - f(b) \forall a, b \in R$.
- The homomorphic image of a ring is a ring.
- The homomorphic image of a commutative ring is commutative ring.
- If f is a homomorphism of a ring R into a ring R' then $\text{Ker } f$ is an ideal of R .
- If f is a homomorphism of a ring r into the ring R' then f is an into isomorphism if and only if $\text{Ker } f = \{0\}$.
- If U is an ideal of a ring R then the quotient ring R/U is a homomorphic image of R .
- Fundamental theorem of homomorphism:
Let R, R' be two rings and $f: R \rightarrow R'$ be an onto homomorphism with Kernel U . Then R' is isomorphic to R/U .
- If $f: R \rightarrow R'$ be a homomorphism and U be an ideal of R then $f(U)$ is an ideal of $f(R)$.
